Advanced Macroeconomics

Section 3 - Growth (II): Ideas, history, geography and institutions

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Section 3 - Growth (II): The Plan

1. Endogenous Growth Theory: key ideas.
2. EG with fixed saving rate and share of R&D.
3. Learning-by-doing: The AK model
4. The Romer (1990) model: endogenous R&D investment.
5. Fundamental determinants of growth

Determinants of innovation

1. Public support for basic research.
   - necessary if innovation is a public good
2. Private incentives for R&D investment
   - requires some excludability
   - rate of return on R&D.
3. Alternative opportunities for talented individuals
   - Baumol (1990); Murphy, Shleifer & Vishny (1991)
4. Learning-by-doing
   - Innovation as a side-effect of economic activity
   - AK models

New growth theory: Key ideas

- Production function for innovation

$$\dot{A}(t) = f(A(t), x(t))$$

x = some measure of R&D efforts.

- *A* is *non-rival* but potentially *excludable*

- Growth *as a result of market-based incentives* requires that innovators enjoy market power.

Simplified EGT model

- 2 sectors: goods production and R&D
- No capital for simplicity
- Fixed share of workforce a_L allocated to R&D
  - $a_L L(t)$ workers in R&D
  - $(1-a_L)L(t)$ workers in goods production.
- Exogenous population growth

$$\dot{L}(t) = nL(t)$$

- Production function for final good:

$$Y(t) = A(t)(1-a_L)L(t)$$

- Production function for new knowledge:

$$\dot{A}(t) = B[a_L L(t)]^{\gamma} A(t)^{\theta}$$

Model dynamics:

- Final good production function implies

$$g_{\frac{Y}{L}}(t) = g_A(t)$$

- Knowledge production function implies

$$g_A(t) = B a_L^{\gamma} L(t)^{\gamma} A(t)^{\theta - 1}$$

  - is there a steady state with constant g_A ?
  - we need to know how g_A evolves over time
  - growth rate of a variable equals derivative of its log wrt to time

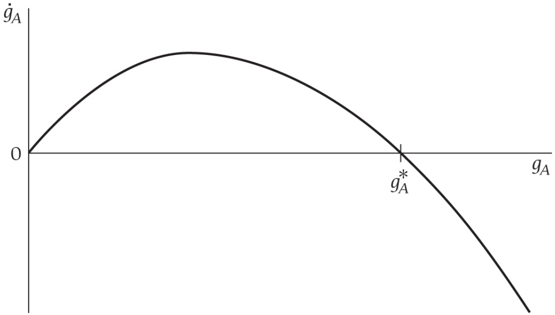
- Taking logs and differentiating wrt time

$$\frac{\dot{g}_A(t)}{g_A(t)} = \gamma n + (\theta - 1) g_A(t) \quad \rightarrow \quad \dot{g}_A(t) = \gamma n g_A(t) + (\theta - 1)[g_A(t)]^2$$

- Value of θ determines the behavior of this model.

$$\dot{g}_A(t) = \gamma n g_A(t) + (\theta - 1)[g_A(t)]^2$$

Case (1): decreasing returns to A ($\theta < 1$)



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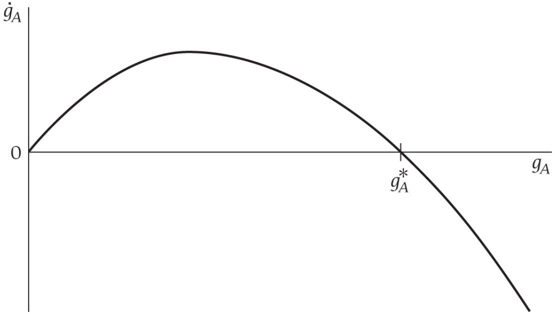
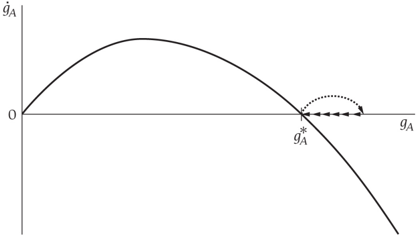
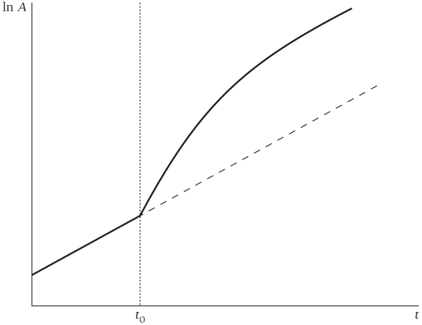


- stable equilibrium
- $g_A^* = \frac{\gamma}{1-\theta} n$;
- no growth effect of a_L and L;
- $g_{Y/L}^*$ depends (positively) on population growth;
- *semi-endogenous growth*.

Effect of a increase in a_L with $\theta < 1$

- $g_A(t) = B{a_L}^{\gamma} L(t)^{\gamma} A(t)^{\theta - 1}$, so g_A initially rises
- But equilibrium growth rate $g_A^* = \frac{\gamma}{1-\theta} n$ unaffected, so effect is temporary.





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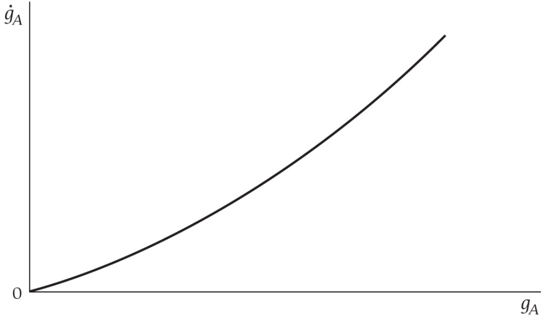
Case (2): increasing returns to A ($\theta > 1$)



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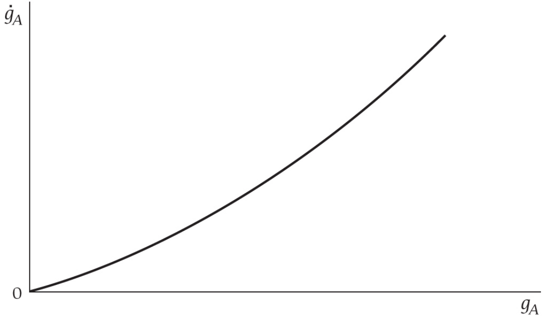

- no equilibrium;
- ever-increasing ('explosive') growth;
- intuition: every marginal addition to A results in a bigger increase in A.

Case (3): constant returns to A ($\theta = 1$)

$$g_A(t) = Ba_L^{\gamma} L(t)^{\gamma}$$
$$\dot{g}_A(t) = \gamma n g_A(t)$$

- Production of new knowledge proportional to its stock
- if $n > 0$, g_A is ever-increasing ('explosive' growth);
- If $n = 0$, g_A fixed and proportional to a_L.
  - no transitions, always in equilibrium
  - fully endogenous growth: growth depends on a_L
  - example of a *linear growth model* ($\dot{A}$ linear in A)

Learning-by-doing: The AK model

Assumptions:

$$Y(t) = K(t)^{\alpha}[A(t)L(t)]^{1-\alpha}; \qquad A(t) = BK(t);$$

$$\dot{K}(t) = sY(t); \qquad L(t) = \bar{L}.$$

Kenneth Arrow

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Dynamics of the model:

$$Y(t) = bK(t) \quad \text{with } b = (B\bar{L})^{1-\alpha}$$

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PS: where have you seen $Y = bK$ and $g_K = sb$ before??

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PS: where have you seen $Y = bK$ *and* $g_K = sb$ *before??*
basically Harrod but with constantly full capacity utilization!

EGT and the *linearity* assumption

- *AK model*
  - o A=BK $\Rightarrow$ g_Y depends on s.
- *EG model with fixed R&D share and* $\theta = 1$
  - o $\dot{A} = \left[B\left(a_L L\right)^{\gamma}\right] A$ $\Rightarrow$ g_Y depends on a_L and L.
- *Romer (1990)* is also a *linear* growth model
  - o $\dot{A} = \left(D L_A\right) A$ $\Rightarrow$ g_Y depends on L.

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- *EG model with fixed R&D share and $\theta = 1$*
  - $\dot{A} = \left[B\,(a_L L)^{\gamma}\right]A$ $\Rightarrow$ g_Y depends on a_L and L.
- *Romer (1990)* is also a *linear* growth model
  - $\dot{A} = (DL_A)\,A$ $\Rightarrow$ g_Y depends on L.
- Linearity $\rightarrow$ stable endogenous growth.
- *The 'trick' of EGT*:
  - if $\dot{A}$ is linear in A, it means that the other factors that multiply A in the knowledge-production function affect the rate of growth of technology (so they will affect growth).
  - $\dot{A} = f(x)A$ $\Rightarrow$ $\frac{\dot{A}}{A} = f(x)$

The Romer model

(a simplified version)

- Output produced from intermediate inputs.
- Technical progress = increasing variety of intermediate inputs.
- Innovation arises from *R&D* investment by private actors.
- Market power: inventor has permanent patent rights.

Assumptions about production

- Production function:

$$Y = \left[\int_{i=0}^{A} L(i)^{\phi} di \right]^{1/\phi}, \quad 0 < \phi < 1$$

  - A continuum of inputs, ranging from 0 to A.
  - $L(i)$ = quantity of input i = labor employed in producing i;
  - Decreasing MP of each input i, but CRS in total inputs amount L_Y.
  - ϕ measures input substitutability

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- When all existing inputs are produced in equal quantities:

$$Y = \left[A \left(\frac{L_Y}{A} \right)^{\phi} \right]^{1/\phi} = A^{\frac{1-\phi}{\phi}} L_Y$$

  - L_Y= workers in inputs production = tot. amount of inputs

Demand for inputs

- Patent-holder hires workers to produce the input associated with her idea
- Inputs then sold to final output producers
- Downward-sloping demand curve for input *i*:

$$L(i) = \left[\frac{\lambda}{p(i)}\right]^{\frac{1}{1-\phi}}$$

$p(i)$ = price of input i.

$1/(1-\phi)$ is the elasticity of demand for inputs.

So $(1-\phi)$ is a measure of market power.

Other key assumptions

- Full employment and fixed labor force:

$$L_A(t) + L_Y(t) = \bar{L}$$

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- Euler equation (from log utility & budget constraint):

$$g_C = \dot{C}(t)/C(t) = r(t) - \rho$$

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- Free-entry condition in the R&D sector:

$$\int_{\tau=t}^{\infty} e^{-r(\tau-t)}\pi(i,\tau)d\tau = \frac{w(t)}{BA(t)}$$

PV of profits from an idea = production cost

The logic of the Romer model

Discount rate (ρ)

↓

Interest rate (r)

↘

Market power of innovators ($1-\phi$)

↙

Discounted returns from R&D activity (R) ← **Population** ($\bar{L}$)

↘ ↙

Number of workers in R&D (L_A)

↓

growth rate of technical knowledge (g_A)

↓

Growth rate of the economy (g_Y)

Solving the model

- $g_Y = \frac{1-\phi}{\phi} g_A + g_{L_Y} = \frac{1-\phi}{\phi} B L_A + g_{(\bar{L}-L_A)}$.
- Steady state → constant L_A.
- Use R&D free-entry condition to infer $L_A^\star$ and thus $g_Y^\star$.

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- Steps:

1. Calculate $\pi(t)$ and $g_\pi = g_\pi(g_W)$
2. Figure out r and g_W
3. Calculate PV of profits from a new idea $R(t)$ using $R(t) = \frac{\pi(t)}{r - g_\pi}$
4. Set PV of profits from idea = production cost, to obtain L_A^* & g_Y^*.

Step 1: find $\pi(t)$ and g_π

- Monopolist patent-holder sets

$$p(i,t) = \frac{\eta}{\eta - 1} w(t)$$

- From demand curve we have:

$$\eta = -\frac{\partial L(i)}{\partial p(i)} \frac{p(i)}{L(i)} = \frac{1}{\phi - 1} \quad \rightarrow \quad p(i,t) = \frac{w(t)}{\phi}$$

- Profits at each point in time:

$$\pi(t) = \frac{\bar{L} - L_A}{A(t)} \left[\frac{w(t)}{\phi} - w(t) \right] = \frac{1-\phi}{\phi} \frac{\bar{L} - L_A}{A(t)} w(t)$$

- Growth rate of profits:

$$g_\pi = g_W - g_A$$

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Step 2: find r and g_W

- All output is consumed and we are assuming constant L_A, so

$$g_C = g_Y = \frac{1-\phi}{\phi} BL_A$$

- Having g_C, we can derive interest rate $r(t)$ from Euler equation:

$$r(t) = \rho + \frac{\dot{C}(t)}{C(t)} = \rho + \frac{1-\phi}{\phi} BL_A$$

- Constant monopoly mark-up implies constant wage share, so

$$g_W = g_Y = \frac{1-\phi}{\phi} BL_A \quad \rightarrow \quad g_\pi = g_W - g_A = \frac{1-\phi}{\phi} BL_A - BL_A$$

Step 3 - Figure out the PV of profits from a new idea

- PV of profits from a new idea:

$$R(t) = \frac{\pi(t)}{r - g_\pi}$$

- From previous steps:

$$\pi(t) = \frac{1-\phi}{\phi}\frac{\bar{L} - L_A}{A(t)} w(t); \quad r = \rho + \frac{1-\phi}{\phi} BL_A; \quad g_\pi = \frac{1-\phi}{\phi} BL_A - BL$$

- Plugging-in:

$$R(t) = \frac{\pi(t)}{r - g_\pi} = \frac{\frac{1-\phi}{\phi}\frac{\bar{L}-L_A}{A(t)} w(t)}{\rho + BL_A} = \frac{1-\phi}{\phi}\frac{\bar{L} - L_A}{\rho + BL_A}\frac{w(t)}{A(t)}$$

Step 4 - Set R(t) = production cost and infer L_A^*

$$\frac{1-\phi}{\phi}\frac{\bar{L}-L_A}{\rho+BL_A}\frac{w(t)}{A(t)}=\frac{w(t)}{BA(t)} \quad \rightarrow \quad L_A^{\star}=(1-\phi)\bar{L}-\frac{\phi\rho}{B}$$

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$$\Downarrow$$

$$L_A^*=\max\{(1-\phi)\bar{L}-\frac{\phi\rho}{B},0\}$$

$$\Downarrow$$

$$g_Y^*=\max\{\frac{(1-\phi)^2}{\phi}B\bar{L}-(1-\phi)\rho,0\}$$

(note: economy always on equilibrium path–no transition dynamics)

(A second look at) The logic of the model

Discount rate (ρ)

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Interest rate (r)

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Market power of innovators $\left(\frac{1-\phi}{\phi}\right)$

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Number of workers in R&D (L_A)

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growth rate of technical knowledge (g_A)

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Growth rate of the economy (g_Y)

Welfare: optimal vs. actual growth

1. *Write PV of lifetime utility as a function of L_A*

$$U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) dt \quad \Rightarrow \quad U = \int_{t=0}^{\infty} e^{-\rho t} ln \left[C(0) e^{g_C t} \right] dt$$

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$$C(t) = Y(t)/\bar{L}; \qquad C(0) = \frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}}; \qquad g_c = g_y = \frac{1-\phi}{\phi} B L_A$$

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$$U = \frac{1}{\rho}\left(ln\frac{\bar{L} - L_A}{\bar{L}} + \frac{1-\phi}{\phi} lnA(0) + \frac{1-\phi}{\phi}\frac{BL_A}{\rho}\right)$$

Welfare: optimal vs. actual growth

2. *Maximize PV lifetime utility w.r.t.* L_A

$$\max_{L_A} U = \frac{1}{\rho}\left(ln\frac{\bar{L}-L_A}{\bar{L}} + \frac{1-\phi}{\phi} lnA(0) + \frac{1-\phi}{\phi}\frac{BL_A}{\rho} \right)$$

$$\downarrow$$

$$L_A^{OPT} = \max\{\bar{L} - \frac{\phi}{1-\phi}\frac{\rho}{B}, 0\}$$

3. *Compare* L_A^{OPT} *with* $L_A^{\star}$

$$L_A^{\star} = (1-\phi)L_A^{OPT}$$

Takeaways:

- Too little R&D ($L_A^* < L_A^{OPT}$);
- more market power for innovators (lower input substitutability ϕ) would increase welfare.

Extensions

- Introducing fixed capital K
  - K produces Y but not $\dot{A}$ –> s has level effect [Romer 1990]
  - but if K produces $\dot{A}$, s can have growth effects.
- Decreasing returns to A in the production of $\dot{A}$
  - –> semi-endogenous growth [Jones 1995]
  - long-run growth depends only on n, while forces affecting L_A have only level effects.
- Quality-ladder models
  - innovation = improvement of existing inputs [Grossman & Helpman 1991; Aghion & Howitt 1992]
  - Similar conclusions.

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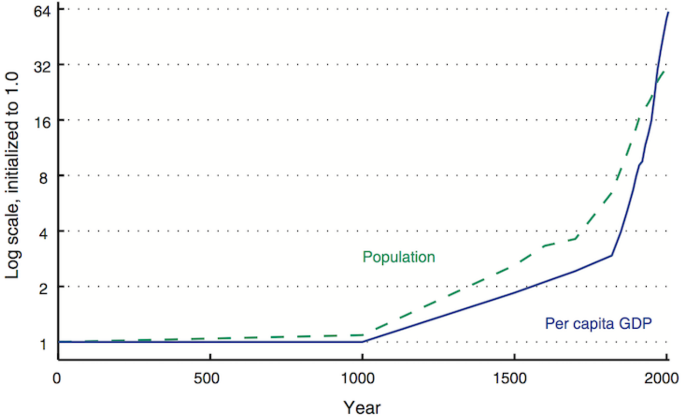
- EGT ignores technological imitation across countries
- L_A & $\bar{L}$ increasing in most countries, but no 'exploding' growth
- P.Krugman: *"too much of [EGT] involves making assumptions about how unmeasurable things affect other unmeasurable things."*
- BUT: EGT might explain growth at a worldwide scale in the very long-run.

GDP per capita & population in US + Europe



(from Paul Romer "The deep structure of economic growth")