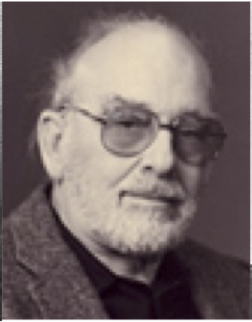
Ramsey-Cass-Koopmans model

Frank Ramsey



David Cass

Tjalling Koopmans

Ramsey-Cass-Koopmans model

- Basically, Solow with microfoundations
- Originally a model of how a central planner would optimally dictate resources allocation [Ramsey, 1928] .
- In 1965 Cass and Koopmans (independently) reinterpret as decentralized equilibrium & connect it to growth theory.
- Our reasons for doing it:
  - Check that Solow model conclusions survive after endogenising the saving rate.
  - Address welfare issues (impossible without preferences).
  - Intro to dynamic optimization, a staple of modern macro.

Assumptions about production

- Production technology and inputs evolution exactly the same as in Solow, but $\delta = 0$ for simplicity.
- $Y = F(K, AL)$; CRS; $f(0) = 0$; $f'(k) > 0$; $f''(k) < 0$;

  $lim_{k \to 0} f'(k) = \infty$; $lim_{k \to \infty} f'(k) = 0$.
- $\frac{\dot{A}}{A} = g_A = g$; $\frac{\dot{L}}{L} = g = n$
- $\dot{K}(t) = Y(t) - \zeta(t)$ where ζ is total consumption.
- *Representative firm* assumption.

Assumptions about households

Large but fixed number of identical households:

- each grows at rate n;
- household members are infinitely lived, forward-looking *and* have perfect foresight into the infinite future;
- they supply 1 unit of L at each point in time and earn wages;
- own K, that they rent to firms, earning K income;
- divide their income between C and I in such a way as to maximize utility over their (infinite) lifetime.
- *representative household* assumption.

The Euler equation

- Assumed households' behavior implies the **Euler equation** – the fundamental driver of this model:

$$g_C = \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}$$

- Higher interest rate induces to postpone consumption, so it contributes *positively* to its growth in time.
- Higher discount rate induces to anticipate consumption, so it contributes *negatively* to its growth in time.
- Let's now study formally how this equation follows from the assumptions of the model...

Household's utility function

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$$

- gives PV of total utility enjoyed by household members over their lifetime.
- C = consumption per person.
- L/H = number of household members.
- ρ = discount rate.
- instantaneous utility $u()$:

$$u[C(t)] = \frac{C(t)^{1-\theta}}{1-\theta} \qquad \theta > 0; \qquad \rho - n - (1-\theta)g > 0$$

Firms & factors' prices

Perfectly competitive firms in single-good economy, therefore

- interest rate: $r(t) = f'[k(t)]$
- wage per unit of eff. labor: $w(t) = \frac{W(t)}{A} = [f(k) - kf'(k)]$

No-Ponzi condition

- Household consumption is constrained by the PV of their wealth:

$$\lim_{s\to\infty} e^{-R(s)}K(s) \geq 0$$

- Or, in 'intensive form' (scaled by AL):

$$\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s) \geq 0$$

- *No-Ponzi condition:* household's asset holdings cannot be negative in the limit.
- Will be satisfied with equality.

Households' dynamic optimization problem

- Household maximizes PV of lifetime utility (intensive form):

$$\text{Max } U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

with $B = A(0)^{1-\theta} \frac{L(0)}{H}$ and $\beta = \rho - n - (1-\theta)g$

- subject to the state equation:

$$\dot{k}(t) = y(t) - c(t) - (n+g)k(t)$$

$$= w(t) - c(t) + (r - n - g)k(t)$$

- and the transversality (no-Ponzi w/ equality) condition:

$$\lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

Solving the household optimization problem

- An optimal control problem, solved by Pontryagin's Maximum Principle.
  - See lecture notes on Keats for details
  - Chiang-Wainwright 'Mathematical Economics' Chapter 20 for an introduction to optimal control.
- Set up the Hamiltonian:

$$H = Be^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta} + \lambda[(r-n-g)k(t) + w - c(t)]$$

- Applying the maximum principle and rearranging to solve for $\frac{\dot{c}(t)}{c(t)}$, we get

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho-\theta g}{\theta} \tag{1}$$

The Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

- This is the **Euler equation** we have seen before, but scaled by AL (intensive form). Same meaning and interpretation.
- $r > \rho \rightarrow$ households postpone consumption $\rightarrow c(t)$ increases in time.
- $r < \rho \rightarrow$ households anticipate consumption $\rightarrow c(t)$ decreases in time.

The dynamics of the economy: c & k

- Dynamics of c (Euler equation):

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

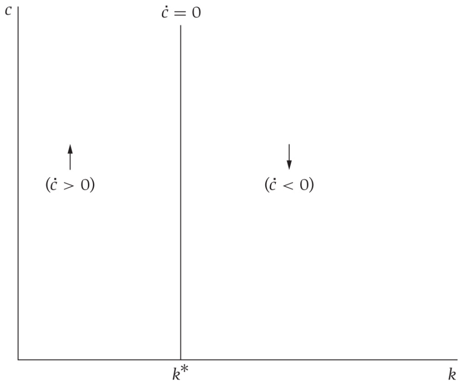
- Dynamics of k (like in Solow but w/o depreciation):

$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$

- intertemporal equilibrium: $\dot{c} = 0$ and $\dot{k} = 0$;
- two variables phase diagram

The dynamics of the economy: consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$



- what's going on in this graph?

The dynamics of the economy: consumption

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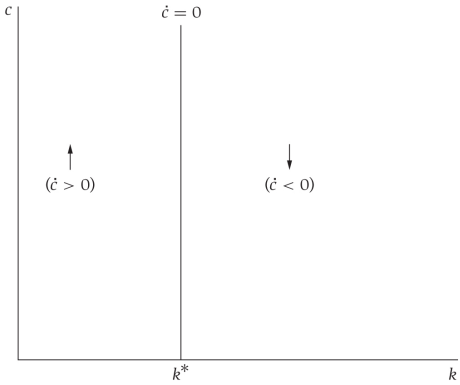
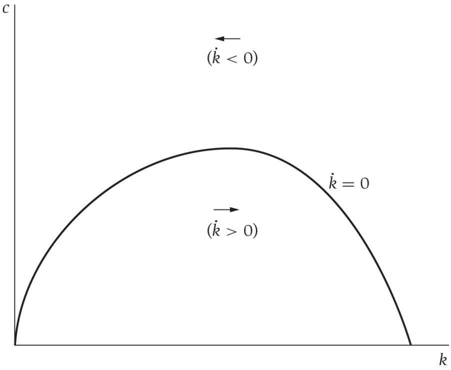


- what's going on in this graph?
- $\dot{c} = 0 \quad \rightarrow \quad f'(k) = \rho + \theta g$
- implicitly defines k^*.
- k^* is unique & independent of c (vertical line).
- $k > k^* \rightarrow f'(k) < \rho + \theta g \rightarrow \dot{c} < 0$
- $k < k^* \rightarrow f'(k) > \rho + \theta g \rightarrow \dot{c} > 0$

The dynamics of the economy: capital stock

$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$



- what's going on in this graph?

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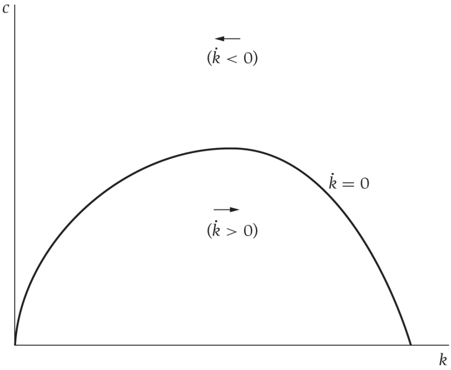
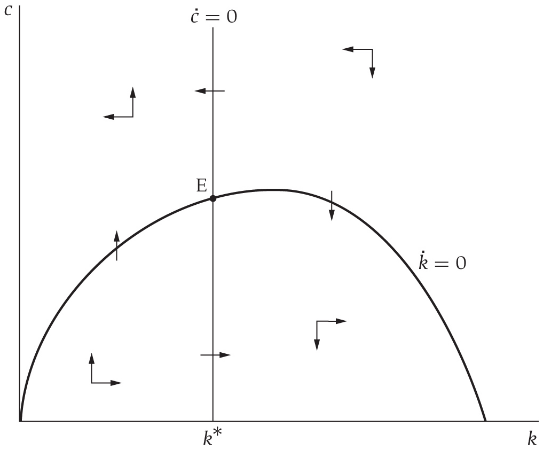


- what's going on in this graph?
- $\dot{k} = 0 \quad \rightarrow \quad c^* = f(k) - (n+g)k$
- c^* U-shaped: increasing in k as long as $f'(k) > (n+g)$.
- $c > c^* \rightarrow$, investment lower than break-even $\rightarrow \dot{k} < 0$.
- $c < c^* \rightarrow$, investment higher than break-even $\rightarrow \dot{k} > 0$.

The dynamics of the economy: phase diagram



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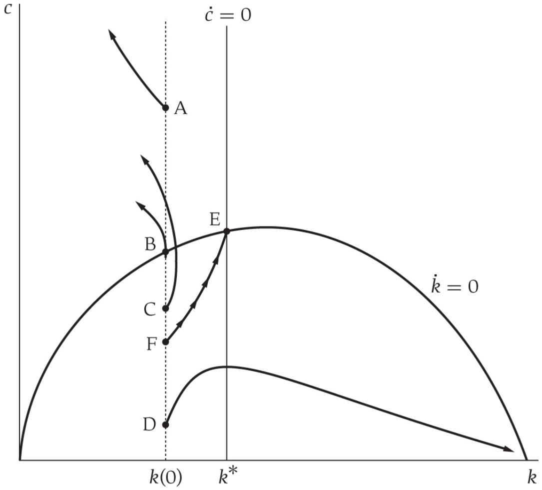

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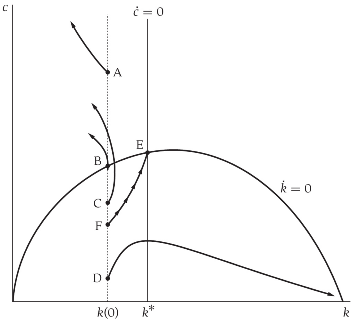


- E = intertemporal equilibrium ($\dot{k} = \dot{c} = 0$);
- given $k(0)$, only $c(0) = F$ is on the 'stable branch' that leads to E;
- $c(0) < F$ leads to zero c and infinite k: not utility-maximizing!
- $c(0) > F$ leads to negative k but positive c: not feasible!
- $c(0) = F$ is the only $c(0)$ that implies

$$\lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

so it is the only feasible and utility-maximizing one.

The saddle path

- For any possible $k(0)$, there is a unique $c(0)$ that satisfies

$$\lim_{s\to\infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

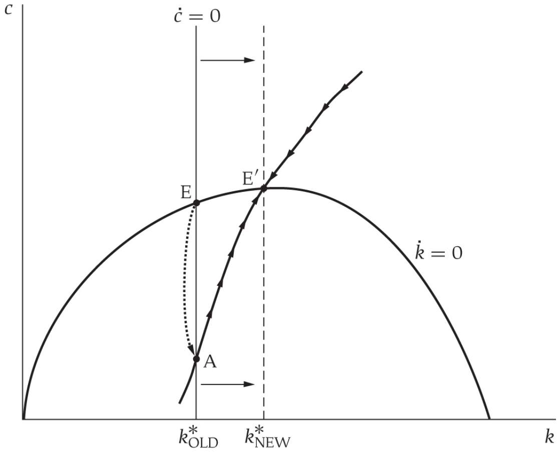
- this $c(0)$ is the one on the 'saddle path'[1] towards steady state.
- all other $c(0)$'s are on unstable trajectories, but are ruled out by the no-Ponzi condition or by intertemporal optimization;
- *saddle-path stable* equilibrium.
- (quite obviously) it is Pareto-efficient.

[1] The reason for this name is the analogy with a marble left on top of a saddle. There is one point on the saddle where, if left there, the marble does not move. This point corresponds to the steady state. There is a trajectory on the saddle with the property that if the marble is left at any point on that trajectory, it rolls toward the steady state. But if the marble is left at any other point, the marble falls to the ground. (Barro & Sala i-Martin, *Economic growth*, 1990).

The balanced growth path

- $\dot{k} = \dot{y} = \dot{c} = 0$
- $g_Y = g_K = g_C = n + g$
- $g_{Y/L} = g_{K/L} = g_{C/L} = g$
  - Same conclusions as Solow model for growth determinants
- here, however, $k^* < k_{GR}$
  - households don't maximize c (as in the *golden rule*), but $PV(c)$.
  - $\rho > 0$ creates a bias towards the present.
- Speed of adjustment faster because s is endogenous.

A fall in the discount rate



- $f'(k) = \rho + \theta g$
- $c^* = f(k) - (n+g)k$

Diamond (1965): The Overlapping Generations (OLG) model

The Overlapping Generations (OLG) model

- Like Ramsey, but there is turnover in population
  - New individuals are born; old individuals pass away.
- No representative household: heterogeneity between old & young.
- Some conclusions change, in particular regarding efficiency and welfare.

OLG model: assumptions about households

- Time is discrete ($t = 0, 1, 2, \ldots$);
- each individual lives for two periods;
- $L_t = (1 + n)L_{t-1}$ individuals born at time t;
- young (1st period):
  - no K
  - supplies 1 unit of L;
  - divides resulting wage between C and S;
- old (2nd period):
  - rents her K (=1st period savings)
  - then consumes $(1 + r)K$

OLG model: assumptions about production

- At each t, old people's K and young people's L are combined to produce Y;
- $Y = F(K_t, A_t L_t)$
- CRS and Inada conditions (as in Solow & Ramsey);
- $\delta = 0$ for simplicity;
- $A_t = (1+g)A_{t-1}$;
- Competitive markets
  - $r_t = f'(k_t)$
  - $w_t = f(k_t) - k_t f'(k_t)$

OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

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Our strategy:

1. U maximization $\rightarrow C$ dynamics (Euler equation);
2. C dynamics $\rightarrow$ dynamics of K & k $\quad\Rightarrow\quad$ k_{t+1} as a function of k_t;
3. set $k_{t+1} = k_t$ to study intertemporal equilibrium & stability.

Consumption dynamics

- Maximization of lifetime-utility...

$$U_t = \ln C_{1t} + \frac{1}{1+\rho} \ln C_{2t+1} \quad \text{with} \quad \rho > -1$$

- ...subject to the budget constraint

$$C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$$

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$\Downarrow$

- f.o.c. imply the Euler Equation:

$$\frac{C_{2t+1}}{C_{1t}} = \frac{1+r_{t+1}}{1+\rho}$$

Discrete-time Euler equation: intuitive derivation

- At the optimal point, a marginal reallocation of C from 1st to 2nd period does not affect utility:

$$\frac{1}{C_{1t}}\Delta C = \frac{1}{1+\rho}\frac{1}{C_{2t+1}}(1+r_{t+1})\Delta C \tag{2}$$

(marginal effect of change in C_1 = marginal effect of change in C_2)

- rearrange as

$$\frac{C_{2t+1}}{C_{1t}} = \frac{1+r_{t+1}}{1+\rho}$$

Discrete-time Euler equation: 'systematic' derivation

- Set the Lagrangian for the utility-maximization problem

$$\mathcal{L} = \ln C_{1t} + \frac{1}{1+\rho} \ln C_{2t+1} + \lambda [A_t w_t - (C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1})]$$

- f.o.c. for C_{1t} and C_{2t}:

$$\frac{1}{C_{1t}} = \lambda; \qquad \frac{1}{1+\rho} \frac{1}{C_{2t+1}} = \frac{1}{1+r_{t+1}} \lambda$$

- Substitute for λ and rearrange:

$$\frac{C_{2t+1}}{C_{1t}} = \frac{1+r_{t+1}}{1+\rho}$$

Consumption dynamics

- Substitute Euler Equation into budget constraint to get

$$C_{1t} = \frac{1+\rho}{2+\rho} A_t w_t$$

- can be rewritten as:

$$C_{1t} = [1-s] A_t w_t$$

$$\text{with} \quad s = 1 - \frac{C_{1t}}{A_t w_t} = \frac{1}{2+\rho}$$

- Constant saving rate, independent on r (but only thanks to log utility!)

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- Constant saving rate, independent on r (but only thanks to log utility!)
- r has both an *income* and a *substitution* effect, and (with log utility) they offset each other.

The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = \frac{1}{2+\rho} A_t w_t L_t$$

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- Given one-good competitive economy, $w_t = f(k_t) - k_t f'(k_t)$, so

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} [f(k_t) - k_t f'(k_t)]$$

The dynamics of the OLG economy

- We now have k_{t+1} as a (implicit) function of k_t:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} [f(k_t) - k_t f'(k_t)] \tag{3}$$

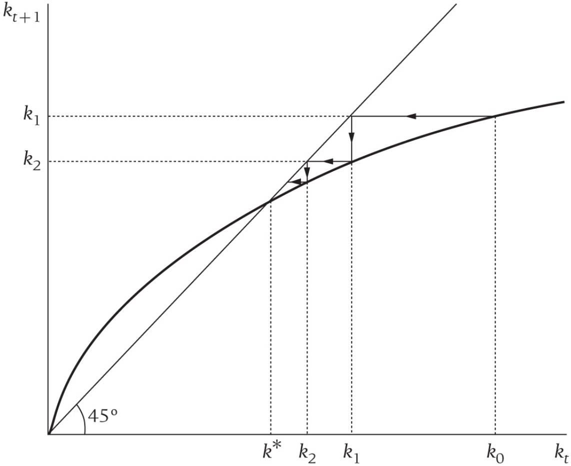
- Assume $f(k) = k^{\alpha}$ with $0 < \alpha < 1$

$$f(k) = k^{\alpha} \Rightarrow \quad f'(k) = \alpha k^{\alpha-1}$$

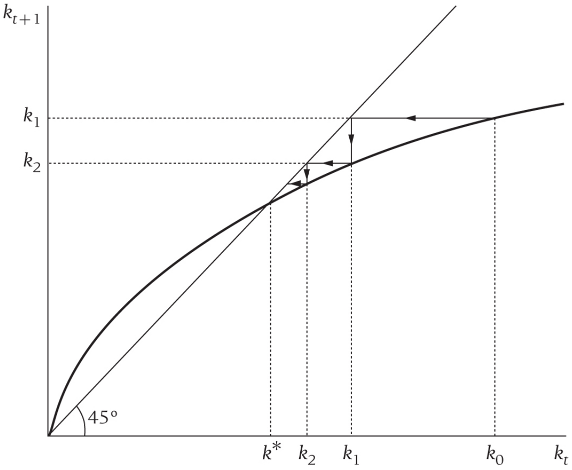
- So the equation of motion for k is:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha} = \beta k_t^{\alpha}$$

The dynamics of the OLG economy



The dynamics of the OLG economy



- Equilibrium: $k_{t+1} = k_t = k^*$;
- decreasing MPK & Inada conditions ensure existence & uniqueness (except $k = 0$);
- dynamically stable;
- steady state à la Solow-Ramsey: constant s and k; Y/L grows at rate g.

The dynamics of the OLG economy

- In intertemporal equilibrium, $k_t = k_{t+1} = k^*$

$$k^* = \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha){k^*}^{\alpha}$$

- Solving for k^*

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

The general case

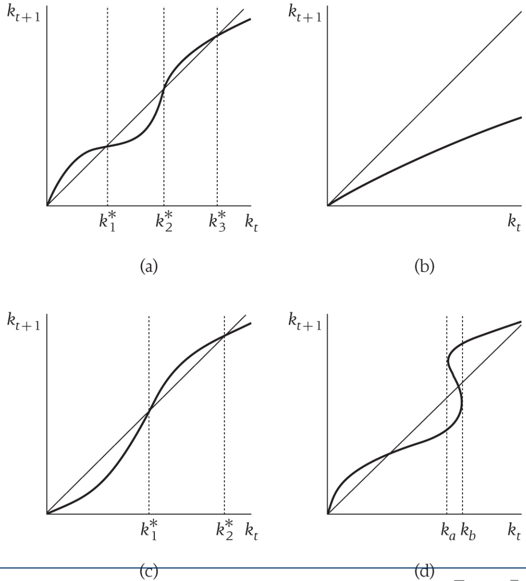
- With more general utility and production functions?
- Equation of motion for k:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r) \frac{f(k_t) - k_t f'(k_t)}{f(k_t)} f(k_t)$$

4 components: [AL_t/AL_{t+1}] [saving rate] [wage share] [Y/AL]

- k_{t+1} depends on k_t through three channels;
- (almost) anything goes.

Diamond: Overlapping generations



- (a) and (c): multiple equilibria (s or W/Y increasing);
- (b): stable zero-output equilibrium (either s or W/Y approach 0 when $k = 0$);
- (d): indeterminacy (s ‘very increasing’ in k_t).

Welfare & dynamic inefficiency

- OLG equilibrium can be Pareto-inefficient ($k^* > k^{GR}$);

Welfare & dynamic inefficiency

- OLG equilibrium can be Pareto-inefficient ($k^* > k^{GR}$);
- Assume $g = 0$, Cobb-Douglass production and log utility:
  - k^{GR} implies $f'(k) = n$.
  - $f'(k^*) = \frac{\alpha}{1-\alpha}(1+n)(2+\rho)$
  - α small $\rightarrow f'(k^*) < n \rightarrow k^* > k^{GR}$
  - This possible *dynamic inefficiency* arises from relaxing the assumption of a finite number of agents.

OLG model: Takeaways

- Same conclusions as Solow/Ramsey on sources of long-run growth;
- but possibility of multiple equilibria and dynamic inefficiency (overaccumulation)

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- Same conclusions as Solow/Ramsey on sources of long-run growth;
- but possibility of multiple equilibria and dynamic inefficiency (overaccumulation)
- Is neoclassical growth theory fragile even within its 'one-good economy with perfect markets' assumptions?
  - However, Barro (1974) shows that a bequest motive (intergenerational altruism) may make OLG practically equivalent to Ramsey (no inefficiencies).
  - Moreover, OLG dynamic inefficiency (over-accumulation) does not seem empirically relevant: too much accumulation??