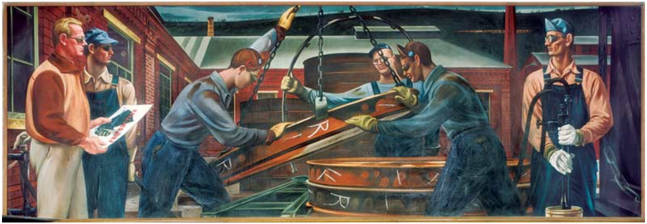

Advanced Macroeconomics

Section 2 - Growth (I): The mechanics of capital accumulation and growth

Daniele Girardi
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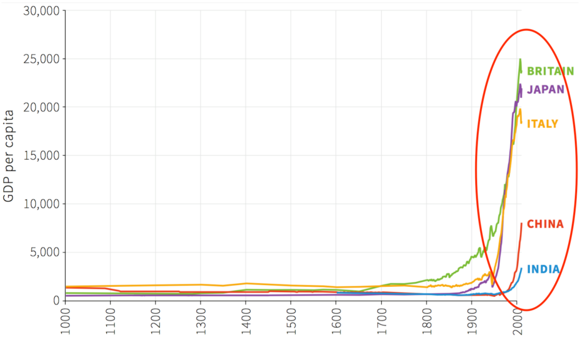
AY 2025-26, Semester I

The hockey stick of history



Section 2: Growth (I)

The Plan

1. Harrod-Domar
2. Solow
3. Ramsey-Cass-Koopmans
4. Diamond’s overlapping-generations (OLG)

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- Models of the *mechanics* of growth
- Interaction of capital accumulation, technological progress & production in a growing economy.
- They can't really explain the hockey stick pattern or the deep fundamental causes of growth.

Dynamic analysis

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    - continuous (except in the OLG model)

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- g_X is a shorthand for $\frac{\dot{X}(t)}{X(t)}$

Intertemporal equilibrium

- Static analysis: equilibrium condition → equilibrium relations.
  - I=S → equilibrium level of Y
  - MRS=MRT → optimal quantity consumed
  - ...
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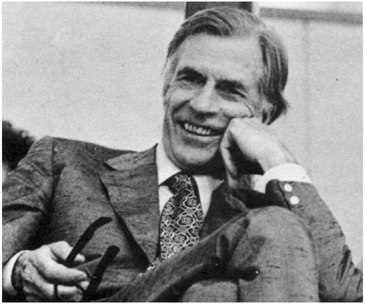
- *Dynamic stability* if over time any deviation from equilibrium tends to (quickly or slowly) disappear.

The Harrod-Domar model

There's an old joke. Two elderly women are at a Catskill restaurant. One of them says, 'Boy, the food at this place is just terrible.' The other one says, 'Yeah I know. And such small portions.'

(Woody Allen, 'Annie Hall')

The Harrod-Domar model

- ‘Grandfather’ of modern growth theory.

- Premise 1: aggregate investment has a dual effect
    1. multiplier effect (demand side)
    2. capacity-creating effect (supply side)

- Premise 2: investment depends on aggregate demand

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- Premise 1: aggregate investment has a dual effect
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- Premise 2: investment depends on aggregate demand
- Main findings:
  - unique equilibrium path: $g_W = sa$ (*warranted rate*)
  - warranted rate does not guarantee full (nor stable) employment
  - instability: economy won’t converge to g_W, except by a fluke

Harrod-Domar: Assumptions

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- Investment rate: $\dot{g}_K(t) = \alpha(u(t) - 1)$ with $\alpha > 0$

Assumptions:

$$Y(t) = C(t) + I(t); \qquad S(t) = sY(t); \qquad Y^*(t) = aK(t); \qquad u(t) = \frac{Y(t)}{Y^*(t)}$$

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)}; \qquad \dot{g}_K(t) = \alpha(u(t) - 1) \qquad \text{with } \alpha > 0$$

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Intertemporal equilibrium (‘warranted’ growth rate):

$$\dot{g}_K(t) = 0 \quad \rightarrow u = 1 \quad \rightarrow g_W = sa$$

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$$g_Y = sa \neq n$$

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- Dynamic instability:

$$g_K = u(sa) > g_W = sa \quad \Rightarrow \quad u > 1 \quad \Rightarrow \quad \dot{g}_K > 0$$
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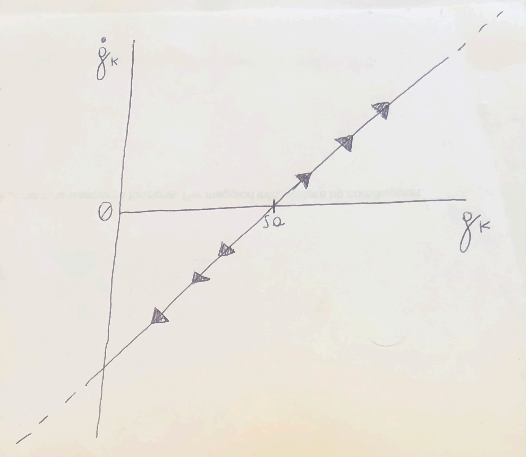
- More formally (by plugging I=S condition into investment function):

$$\dot{g}_K = \alpha \left[\frac{g_K}{g_W} - 1 \right]$$

→ *When the growth rate is above equilibrium ($g_K/g_W > 1$), it will tend to increase even more (and vice versa)*

$$\dot{g}_K = \alpha \left[\frac{g_K}{g_W} - 1 \right] \quad \text{with } \alpha > 0$$



- Phase diagram.
- positive slope means instability