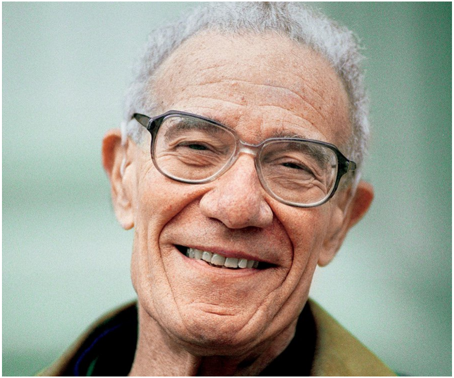
The Solow model



Solow growth model

Key premises:

- *neoclassical* production function
- Say's law: full employment at all times.

Main implications:

- stable steady-state with $g_Y = n + g$
- saving rate determines output level but not growth rate
- K accumulation cannot explain long-run growth or cross-country income differences.

Production function

- One-good economy
- 4 variables: Y, K, L, A.
- Say's law: full employment of L & K at each t.
- Neoclassical aggregate production function

$$Y(t) = F[K(t), A(t)L(t)]$$

  - AL: labor-augmenting technological progress.

Constant returns to scale (CRS)

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- *Intensive form* of the production function:

$$\frac{Y}{AL} = F(\frac{K}{AL}, \frac{AL}{AL}) = F(\frac{K}{AL}, 1)$$
$$\Downarrow$$

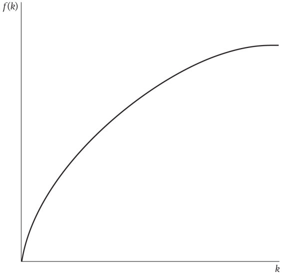
$$y = f(k)$$

with $k = \frac{K}{AL}$, $y = \frac{Y}{AL}$ and $f(k) = F(k, 1)$

Other assumptions about the production function

$$f(0) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0$$

$$lim_{k \to 0} f'(k) = \infty, \qquad lim_{k \to \infty} f'(k) = 0$$



Evolution of production inputs

$$\dot{L}(t) = nL(t) \rightarrow g_L = n$$

$$\dot{A}(t) = gA(t) \rightarrow g_A = g$$

$$\dot{K}(t) = sY(t) - \delta K(t), \qquad 0 < s \leq 1$$

The dynamics of the model

- Strategy: focus on $k = \frac{K}{AL}$
- Take the derivative of k wrt time

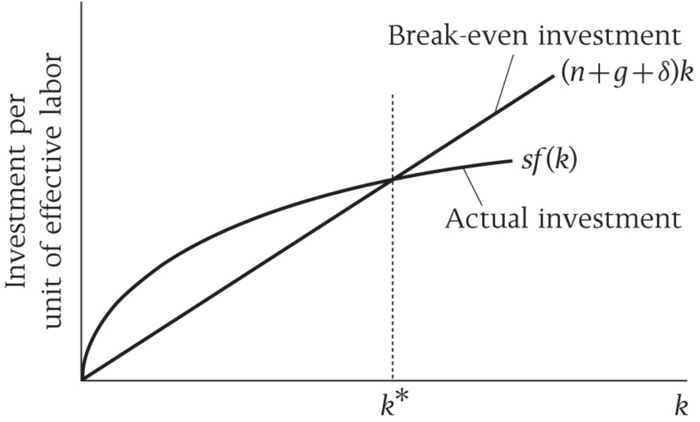
$$\dot{k}(t) = \frac{d(K/AL)}{dt} = \frac{\dot{K}}{AL} - \frac{K}{(AL)^2}(A\dot{L} + \dot{A}L) = \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A}$$

- using $k = \frac{A}{AL}$, $y = \frac{Y}{AL}$ & the assumptions about inputs:

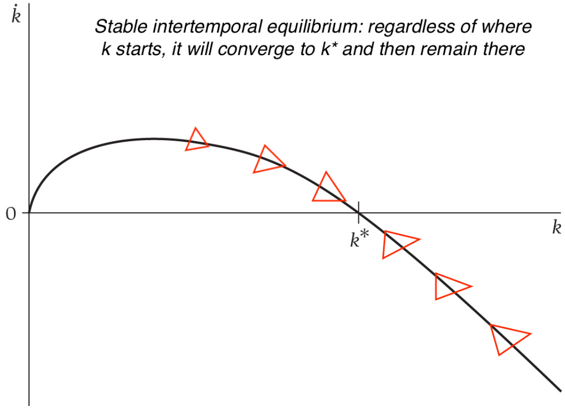
$$\dot{k}(t) = sf[k(t)] - (n + g + \delta)k(t)$$

change in k = investment − breakeven investment

Actual vs. break-even investment



Phase diagram



The steady state

In the intertemporal equilibrium...

- by assumption, $g_L = n$ and $g_A = g$;
- $K = ALk \rightarrow g_K = n + g$
- $Y = ALf(k) \rightarrow g_Y = n + g$
- $\frac{K}{L} = Ak \rightarrow g_{\frac{K}{L}} = g$
- $\frac{Y}{L} = Af(k) \rightarrow g_{\frac{Y}{L}} = g$

balanced growth path: all variables grow at constant rates.

Other things we want to know:

1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
3. Size of the effect of an increase in s (*how big*)
4. Speed of convergence: *how long* does transition take?

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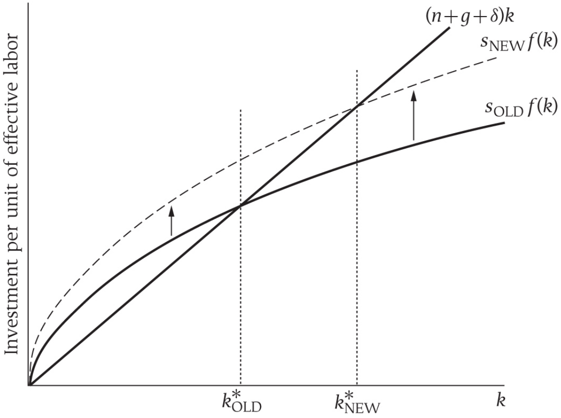
An increase in the saving rate



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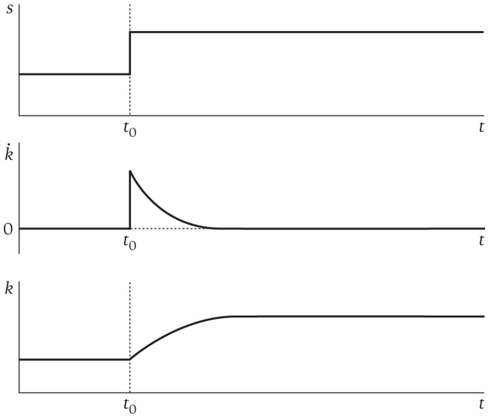
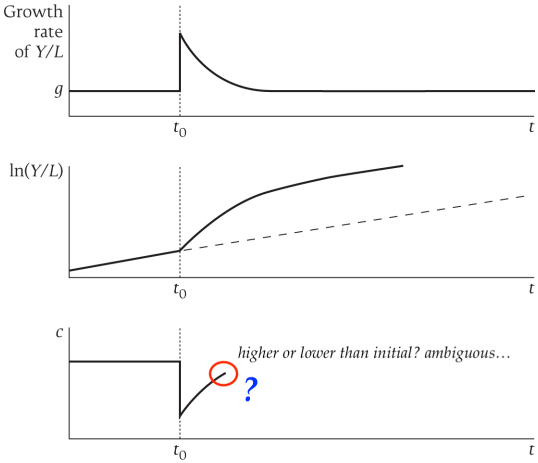


An increase in the saving rate

$$Y/L = Af(k)$$

$$c = f(k)(1-s)$$



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- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s, n, g, \delta)$

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$$f'(k^*) = (n+g+\delta)$$

- golden-rule k^*
- characterized by $MPK = (n+g+\delta)$.
- but no reason for s to be exactly at the level which implies the golden-rule k^*!

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$$\Rightarrow \frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta)-sf'(k^*)}$$

- convert this to an elasticity, use $sf(k^*) = (n+g+\delta)k^*$ and rearrange to get:

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

where $\alpha_K(k^*) = \frac{k^*}{f(k^*)}f'(k^*)$ is the elasticity of y w.r.t. k.

Effect of the saving rate on output

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

$$\text{Assume} \quad \alpha \approx 1/3 \quad \Rightarrow \quad \frac{s}{y^*}\frac{\delta y^*}{\delta s} \approx \frac{1}{2} \tag{1}$$

- A 1 percent increase in s increases y^* by 0.5 percent;
- ex: raising s from 0.20 to 0.22 (+10%) increases y^* by 5%;
- significant but not very big.

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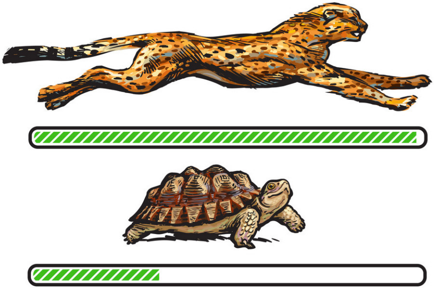
Speed of convergence

- *How fast* will k reach k^* when starting out of equilibrium?



Refresher: Taylor approximations

- *Taylor's theorem:* any (continuously differentiable) function $\phi(x)$ can be approximated, around a point x_0, by a n-th degree polynomial.
- n-th degree Taylor approximation around x_0:

$$\phi(x) = [\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n] + R_n$$

(R_n = remainder)

- linear approximation around x_0:

$$\phi(x) \approx \phi(x_0) + \phi'(x_0)(x - x_0)$$

Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around k^*:

$$\dot{k} \approx \left[\frac{\partial \dot{k}(k)}{\partial k} \bigg|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda (k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

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- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$
- $\lambda = -\frac{\delta \dot{k}}{\delta k}|_{k=k^*} = (n + g + \delta)(1 - \alpha_K)$
- Bottom line: for plausible parameters, convergence is not fast.
- eg: $(n + g + \delta) = 6\%$ and $\alpha_K = 1/3 \rightarrow \lambda = 0.04$

Solow model: Takeaways

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3. Two sources of cross-country variation in Y/L: s and A.
   - BUT implausibly huge differences in s would be needed to produce sizable differences in Y/L.
4. Technology (A) is the only possible explanation of vast cross-country differences in Y/L.

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- Subsequent developments of neoclassical growth theory address these issues.

Solow model: more radical criticisms

- Overcomes Harrodian instability by assuming it away
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- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.
- Relies on a fictional aggregate production function $Y(K, AL)$.
  - In reality, many production processes and many types of inputs
  - They do not add up to an aggregate production function with the properties assumed by Solow model (Cambridge capital controversy)
  - One-good economy: *'Venerable Solow may make peculiar assumptions, but he never makes a mistake'* (A. Sen, 1974)
  - Recent discussion of the problem: Baqaee & Fahri (2019).
- Subsequent developments of neoclassical growth theory do *not* address these issues.

Growth accounting

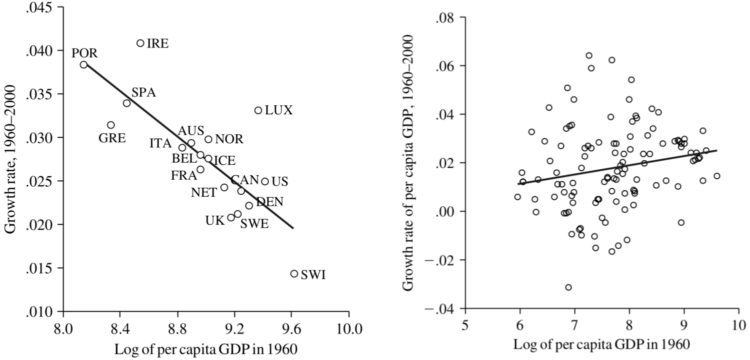
Given the production function, we can decompose $g_{Y/L}(t)$ into:

$$g_{Y/L} = \alpha_K g_{K/L} + R$$

- α_K estimated using P/Y;
- residual R interpreted as the contribution of unobservable technological progress.
- But it could also be every other thing that the model leaves out!

Convergence regressions

- Do poor countries catch up?
- Solow model suggests convergence, if A non-excludable.
- Empirical test:
$$\Delta ln(Y/L)_{i,1} = \alpha + \beta ln(Y/N)_{i,0} + \epsilon_i$$
- with $t = 0$ and $t = 1$ usually quite apart in time (40/50 years).
- $\beta = -1$ = perfect convergence
- Evidence from 1960-2000: some convergence among core-OECD countries ($\beta \approx -1$), but little or no convergence overall ($\beta \approx 0$).



(a) 18 original OECD members (b) world (114 countries)

Figure: International evidence on convergence: 1960 income and subsequent growth