The Harrod-Domar model*

Daniele Girardi†

The Harrod-Domar model provided the initial 'impulse' which gave birth to modern theories of economic growth.[1] Indeed, the Harrod model is the precursor of all types of growth models, including the neoclassical ones on which this course will focus, but also alternative models like the post-Keynesian ones. It is possible (and in fact quite insightful) to see different subsequent growth models as different ways to solve the problems highlighted by Harrod.

1 Overview

When developing his seminal theory, in the 1940s, Harrod aimed to model the process of capital accumulation and economic growth. Keynesian theory, which had just born, provided a theory of the determination of the level of output in the short run. Harrod wanted to extend the Keynesian framework by exploring its dynamic implications, to explain the *growth* of output over time.

The Harrod model has two basic premises. The first is that any change in aggregate investment has a dual effect: a demand-side effect and a supply-side one. On the demand side, changes in investment determine changes in output through the *multiplier effect*. To invest, firms must buy capital goods (equipment, structures, machinery) from other

*Lecture notes for Advanced Macroeconomics (7SSPN402), Fall 2023.

†Lecturer, Department of Political Economy, King's College London.

[1]See Harrod (1939) and Domar (1946). In what follows we will just call it the Harrod model, for brevity and because Harrod apparently had the idea first. If you are interested in the 'history of economics' aspect of this – including the debate on 'what Harrod really meant', and whether he would be happy with the currently prevailing interpretation of his work – you can take a look at a series of interesting papers by Daniele Besomi. A nice presentation of the Harrod model (on which these notes partly draw) can be found in Skott (1989, p. 18). A pared-down but helpful exposition focused on the mathematics of the model, based on the Domar version, can also be found in Section 16.4 of the fourth edition of Chiang's 'Fundamental Methods of Mathematical Economics' (which was a source of inspiration for these notes too).

firms, thus stimulating the economy and increasing aggregate production and income. Therefore, on the demand side, the higher the rate of investment, the higher demand and production. At the same time, on the supply side, investment adds to the capital stock and thus increases the productive capacity of the economy (also called *potential output*). The higher the rate of investment, the higher the subsequent potential output of the economy. Harrod wanted to study the interaction between these two effects.

The second premise concerns the determinants of aggregate investment. Harrod assumed that investment depends on aggregate demand: firms invest more when they experience strong demand for their products, and invest less when demand is weak. Firms invest to expand their productive capacity, and they want to do so if demand for their products is growing. Therefore, the higher the growth rate of demand and output, the higher the investment rate. This positive relation between output and investment is also called the *accelerator* effect.

Harrod found that the combination of these ingredients, in a stylized model of a closed economy with no government, results in a dynamic economic system with wildly unpleasant properties. The dynamic system has an equilibrium growth rate (the 'warranted' growth rate, as Harrod calls it) – a unique rate of growth which is compatible with the optimal rate of utilization of capital, and therefore does not induce further changes. However, this equilibrium growth rate has two disturbing properties: first, it does not guarantee full labor employment; second, it is not stable, meaning that any deviation from equilibrium will be amplified in a self-reinforcing explosive or implosive pattern.

The resulting economic picture is reminescent of a classic Woody Allen joke (from Annie Hall), in which two elderly women are at a restaurant. One of them says, 'Boy, the food at this place is just terrible'. The other one replies, 'Yeah I know. And such small portions.' Similarly, in the Harrod model the equilibrium growth rate does not guarantee full (nor stable) employment, and is not likely to be reached anyway

2 A brief brush-up about dynamic analysis

Before delving into the Harrod model and the other growth theories we will study in this course, it is useful to review some basic notation and concepts used in this type of dynamic economic models. If you are familiar with basic dynamic analysis, you can skip this Section.

This type of model aims to explain the dynamic evolution of the economy, and therefore the main variables of interest are not constant, but change over time. Time is represented by the variable t. If a variable X changes over time, we write $X(t)$. This means that X is a function of time. $X(t)$ is the value of X at time t. Note that here t is a continuous variable (in other models, instead, time can be discrete).

Given that X is a function of time, we can take the derivative of X with respect to time, $\frac{dX}{dt}$. Because in this type of dynamic models the derivative with respect to time is used pretty often, practitioners came up with a special symbol for this derivative. Specifically, we use a dot over a variable as a shorthand for its derivative with respect to time. That is, $\dot{X}(t)$ is a shorthand for $\frac{dX}{dt}$.

If you think about the meaning of a derivative with respect to time, you will realize that $\dot{X}(t)$ represents the *rate of change* of X, while $\frac{\dot{X}(t)}{X(t)}$ is the *growth rate* of X. We use g_X as a shorthand for the growth rate of X. That is, g_X is a shorthand for $\frac{\dot{X}(t)}{X(t)}$.

When economists study a model of the economy, they typically want to understand its *equilibrium*. In a static model, which aims to explain the *level* of output of the economy at a given point in time, we usually find the equilibrium of the model by imposing some condition that the equilibrium must satisfy. We then figure out what is the level of output (and of the other relevant variables) consistent with the equilibrium condition. This equilibrium level of output will be a function of the exogenous parameters of the model.

For example, in a basic static model of short-run output determination like the one you might have studied in intermediate macroeconomics, you impose the equilibrium condition that supply must equal demand, and through some algebra you figure out that equilibrium output equals autonomous demand divided by the propensity to save. Or in a basic microeconomic model, we impose the condition that the marginal rate of transformation must be equal to the marginal rate of substitution, to derive the optimal consumption choice of an individual. In other words, in a static model we use some equilibrium condition to derive the equilibrium relations of the model. This is called a *static* equilibrium.

Growth theory, however, is not static, but *dynamic*. When we study a dynamic economic model, we need to understand its dynamic equilibrium, also called *intertemporal equilibrium*. An intertemporal equilibrium is a situation in which the key endogenous variables of the model are constant over time, with no endogenous reason for change. Often, in economic models, the key variable that stays constant over time is a growth rate itself. For example, in the intertemporal equilibrium of a growth model, it is usually the growth rate of output that stays constant over time, not its level. This type of intertemporal equilibrium is called a *steady state*. Often, in a steady state, *ratios* between variables (for example, the ratio of fixed capital to output) remain constant in time, making it useful to focus on them when analysing the model.

Usually, after figuring out the intertemporal equilibrium of a dynamic model, we want to assess *dynamic stability*. Dynamic stability tells us whether the model is stable or not. A dynamic model is stable if any deviation from equilibrium tends to disappear (quickly or slowly) over time. A dynamically stable model always converges to its intertemporal equilibrium, no matter what the initial values of the model parameters are. A dynamically unstable model, instead, does not converge towards the intertemporal equilibrium.

You can think of the intertemporal equilibrium as a special place where the model can go. This special place has a unique feature: when the model arrives there, it stops moving and stays there forever, unless some external (exogenous) shock displaces it. Dynamic stability means that if the model starts in any other place, it will move towards the intertemporal equilibrium place.

3 The Model

The Harrod model is quite simple, at least mathematically. Assume an economy that produces only one good. Each produced unit of the good can either be consumed or invested. If it is invested, it means that it is accumulated as fixed capital for subsequent production. The economy is closed (no international trade) and there is no public sector. We also assume no technological progress and no depreciation (fixed capital is eternal) for simplicity.

The number of units of the good that are produced in the economy at time t is denoted as $Y(t)$. That is, $Y(t)$ is the output level. The amount of the good that is used for consumption is $C(t)$. The amount of the good that is not consumed is called *savings*, denoted as $S(t)$.

The saving rate s is the fraction of total output that is saved. It is equal to $s = S(t)/Y(t)$. The saving rate s is assumed to be fixed and constant over time (this is why we can write simply s, instead of $s(t)$). Total savings are thus equal to $S(t) = sY(t)$.

In order to produce, firms use labour and fixed capital, in fixed proportions. Therefore, the productive capacity of the economy depends on the stock of fixed capital, which we call $K(t)$. However, fixed capital can be utilized more or less fully: firms can use production plants less intensely when they face low demand, or more intensely when they face higher demand.

When firms utilize their capital stock at the planned (optimal) rate, they achieve the optimal ratio of output to capital, which we call a. The optimal output-capital ratio a is exogenously given and constant in time, determined by the available technique of production.

To formalize these assumptions, we can write the *production function* $Y^*(t) = aK(t)$. In this formula, Y^* is potential output: the level of output that would be produced if the rate of utilization of the capital stock was the planned one. Given the optimal output-capital ratio, potential output is a positive function of the capital stock. The more capital stock the economy has, the more it can produce.

However, the actual level of output $Y(t)$ does not need to be equal to potential output $Y^*(t)$. This is because we have assumed that productive capacity is flexible, at least in the short-run: based on the realized level of demand for their product, firms can end up either under-utilizing their productive capacity or over-utilizing it.

The rate of capacity utilization $u(t)$ is defined as $u(t) = \frac{Y(t)}{Y^*(t)}$. Therefore, $u = 1$

corresponds to the normal (optimal) rate of utilization, while $u > 1$ implies a 'heated' economy in which demand outpaces productive capacity and firms over-utilize their machines, and $u < 1$ implies a depressed economy with idle (under-utilized) machines.

Investment is denoted as $I(t)$. Given that investment is the addition to the existing capital stock, it is defined as $I(t) = \dot{K}$. The growth rate of the capital stock, also called *investment rate*, is equal to $g_K = \frac{\dot{K}}{K} = \frac{I}{K}$.

Investment decisions are driven by demand dynamics, consistent with the accelerator principle described earlier. The simplest way to represent this, in this context, is to write an investment function in which changes in the investment rate (g_K) depend on the utilization rate, like the following:

$$\frac{dg_K(t)}{dt} = \dot{g}_K(t) = \alpha(u(t) - 1) \qquad \text{with } \alpha > 0 \tag{1}$$

When experiencing a shortage of productive capacity relative to demand ($u > 1$), firms will increase their investment rate to make productive capacity grow faster. When experiencing under-utilization of their productive plants ($u < 1$), firms' investment rate will decrease. If capital is utilized at just the right rate, firms will be satisfied with their investment rate and will not change it ($\dot{g}_K(t) = 0$).

3.1 The warranted rate of growth

At all points in time, realized savings must equal investment ($I = S$). The realized growth rate of the capital stock g_K must be consistent with this static equilibrium condition. We thus have:

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)} = \frac{S(t)}{K(t)} = s\frac{Y(t)}{K(t)} = s\frac{Y^*(t)}{K(t)}\frac{Y(t)}{Y^*(t)} = sa(u(t)) \tag{2}$$

In words, the investment rate is equal to the product of the saving rate, the optimal output-capital ratio and the rate of capacity utilization.

This economy is in a dynamic (or *intertemporal*) equilibrium when the investment rate is stable over time: $\dot{g}_K = 0$. A situation where g_K is stable over time is called the equilibrium growth path – or 'warranted' rate, as Harrod called it.

Equation 1 tells us that in order to have $\dot{g}_K = 0$, we need to have $u = 1$. Plugging this into equation 2, we see that the warranted rate (g_W) is equal to

$$g_W = sa \tag{3}$$

On such an equilibrium path, the actual output-capital ratio would stay constant (and equal to its optimal value a), which implies that g_W is the equilibrium growth rate of both capital stock and output: in equilibrium $g_Y = g_K = g_W$.

As long as the economy grows at the warranted rate, aggregate demand and productive capacity grow at the same pace, and the rate of utilization of the capital stock stays stable at its optimal value ($u = 1$). The warranted path can be seen as a rational expectations equilibrium: firms' investment plans turn out to be based on correct demand expectations, allowing them to reach precisely their target rate of utilization.[2]

To sum up, the Harrod model has a dynamic equilibrium, in which output and the capital stock grow at a constant rate. The equilibrium growth rate (the 'warranted' rate of growth) is equal to the product of the saving rate and the optimal output-capital ratio. The higher the propensity to save (as measured by the saving rate s), the higher the 'warranted' growth rate. The higher the productivity of capital (as measured by the output-capital ratio a), the higher the 'warranted' growth rate. This intertemporal equilibrium is a steady state, because the growth rate of capital and output is constant. As long as the rate of growth is equal to its equilibrium value, the economy continues to grow along this dynamic equilibrium path.

3.2 Warranted vs. natural growth rate

One major implication of the Harrod model is that there is no reason for the warranted rate to guarantee full or stable employment. Let us see why.

Assume that the labor force grows at some given rate n. With no technical progress and a given technique of production, employment is proportional to output, and grows at the same rate. A necessary condition for full employment is thus that output grows

[2]Harrod describes the warranted rate as *"that rate of growth which, if it occurs, will leave all parties satisfied that they have produced neither more nor less than the right amount"* (Harrod, 1939, p. 16).

at rate n. This is actually necessary not only for full employment, but for the unemployment rate to be stable at all: if the growth rate of output and population do not coincide over long periods of time, this will lead the economy to eventually run out of labor (if $g_Y > n$) or to an ever-rising unemployment rate (if $g_Y < n$). For this reason, n is called the 'natural' rate of growth of this economy.[3]

The problem is that the condition for dynamic equilibrium ($g_K = g_Y = sa$) is completely independent from the condition for a stable unemployment rate ($g_Y = n$). s, a and n are all exogenous in this model, and they come from different sources: there is no reason for the economy to fully employ labor, and not even to display a stable unemployment rate. Even if we could guarantee that the economy converged to the warranted growth path (and we will see that we actually can't), the pattern of the unemployment rate would in all likelihood be a concerning one.

3.3 Harrodian instability

$g_K = g_W = sa$ is a dynamic equilibrium: as long as the growth rate is exactly equal to g_W, we will have $u = 1$ and so the accumulation rate (governed by equation 1) will stay constant at its warranted rate.

But what happens out of equilibrium? Will the system tend to converge towards the warranted rate? Quite the contrary. In this model, if the economy is growing faster than the equilibrium rate, its growth rate will increase ever more, quickly approaching infinity. If the economy is growing slower than the equilibrium rate, it would collapse towards zero production. Let us see why.

Imagine a situation in which $g_K > g_W$: the investment rate is above its equilibrium value. This implies $u > 1$ (overutilization of productive capacity). So firms will *increase* their investment rate further, in order to try to address the shortage of productive capacity that they are experiencing. This will increase even more the discrepancy between g_K and g_W, between Y and Y^* and between u and 1. In turn, this will lead to a further increase in the investment rate, in an explosive pattern that would make the investment rate and the rate of utilization tend to infinity. The intuition is that

[3] With technological progress making labor productivity grow at some rate m, the natural rate of growth would be $n + m$.

each individual firm is expanding its capacity to meet the excess demand; collectively, however, this results in a multiplier effect of aggregate investment which is stronger than the capacity-generating effect, so that excess demand grows even faster.

Similarly, if you start from a situation of under-utilization ($g_K < g_W$, which implies $u < 1$), this will lead firms to make the *wrong* kind of adjustment, reducing their investment rate further and further. The investment rate and the utilization rate would follow a path of collapse.

A more formal way to see the instability problem is to use equations 1 and 2 to obtain the following relation

$$\dot{g}_K = \alpha[\frac{g_K}{g_W} - 1] \tag{4}$$

This implies that the change in the growth rate is a *positive* function of the discrepancy between the actual and the warranted rate. When the growth rate is above equilibrium ($\frac{g_K}{g_W} > 1$), it will tend to increase even more. When it is below equilibrium, it will tend to decrease further.

Stability of the equilibrium would require $\alpha < 0$: we would need firms to decrease their investment rate whenever it is above equilibrium, and to increase it when it is below. But this cannot happen: no firm wants to increase its investment when its productive capacity is under-utilized.

3.4 Takeaways

What to take away from this? One possible interpretation is that there must be something wrong with the Harrod model: we do not observe this kind of explosive instability in the real world, and we observe relatively stable unemployment rates (at least most of the time). This interpretation underlies the subsequent development of mainstream neoclassical growth theory. As we will see, the neoclassical growth model assumes that the economy is always at full employment (according to the so-called "Say' s law"), with investment passively adapting to savings, and that the optimal output-capital ratio is flexible. In this way the utilization rate is *by assumption* always at its optimal value

($u = 1$), and the parameter a adjusts to ensure that $g_K = sa = n$.[4]

Another possible interpretation is that the Harrod model captures a fundamental source of instability that comes from the (private) goods market of the economy. But it leaves out very important parts of the economy, like the labor market, monetary policy, the fiscal sector and the external sector. Stabilizing forces could come from (some of) these other parts of the economy, and this may be why we do not generally observe extreme instability of the Harrodian type.

References

Domar, E.D. (1946). "Capital Expansion, Rate of Growth, and Employment". In: *The Economic Journal* 14.2, pp. 137–147. DOI: 10.2307/1905364.

Harrod, R.F. (1939). "An Essay in Dynamic Theory". In: *The Economic Journal* 49.193, pp. 14–33. DOI: 10.2307/2225181.

Skott, P. (1989). *Conflict and effective demand in economic growth.* Cambridge University Press.

[4] This interpretation underlies also some subsequent 'post-Keynesian' proposals, in which it would be income distribution (the saving rate) which adjusts to ensure equilibrium.